JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **219,** 331–343 (1998)
ARTICLE NO. AY975813

Weak Solutions to a Penrose-Fife Model with Fourier Law for the Temperature

Ph. Laurençot

Institut Elie Cartan-Nancv. Université de Nancv I. BP 239.

Received April 30, 1996

We prove global existence in time of weak solutions to the Penrose–Fife model when the heat flux is given by the Fourier law, and the homogeneous free energy density by the classical double well potential. The main feature of these weak solutions is the almost everywhere positivity of the absolute temperature throughout time evolution.

1. INTRODUCTION

In [10], O. Penrose and P. C. Fife proposed a thermodynamically consistent phase-field model to describe the kinetics of phase transitions in binary systems. This model involves an order parameter ϕ (which is the state variable characterising the phase) and the absolute temperature θ ($\theta > 0$). Under some structural assumptions on the free and internal energy densities, they derive the following system of partial differential equations,

$$\tau\phi_t - \xi^2 \Delta\phi + s'(\phi) = \frac{\lambda'(\phi)}{\theta} \qquad \text{in } \Omega \times (0, +\infty), \tag{1.1}$$

$$c\theta_t + \operatorname{div}\left(\kappa(\phi, \theta)\nabla\left(\frac{1}{\theta}\right)\right) = \lambda'(\phi)\phi_t \qquad \text{in } \Omega \times (0, +\infty), \tag{1.2}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(1 \leq N \leq 3)$ with smooth boundary Γ, τ, ξ, and c are positive real numbers, s is a double well potential, and λ is a quadratic polynomial.

0022-247X/98 $25.00

The heat flux $\mathbf{q}$ is here given by

$$\mathbf{q} = \kappa(\phi, \theta)\nabla\left(\frac{1}{\theta}\right), \tag{1.3}$$

for some nonnegative function $\kappa(.,.)$.

When investigating existence of global in time solutions to (1.1)–(1.2), the first mathematical difficulty encountered arises from the presence of the inverse temperature in the equations. More precisely, given a positive initial temperature distribution, a natural question is whether the temperature distribution will remain positive or not through time evolution (so that the inverse temperature makes sense or not). At this point, let us mention that no maximum principle, which would guarantee the positivity of temperature, seems to be available for (1.1)–(1.2).

Several authors have investigated existence of global in time solutions to (1.1)–(1.2) when $\kappa(\phi, \theta) = \text{const.} > 0$ [13, 12, 6, 7, 5]. In this case, there is a balance in (1.2) between the positive power of θ in the time derivative, and the negative power of θ in the heat flux $\mathbf{q}$, which yields global existence in time of solutions (ϕ, θ) to (1.1)–(1.2) with positive temperature θ. Some extensions of the above results to more general functions $\kappa(\phi, \theta) = \kappa_0(\theta)$ may be found in [8, 3], but they still rely on the above mentioned balance in (1.2).

However, in their original model, O. Penrose and P. C. Fife assume that the heat flux $\mathbf{q}$ is given by the Fourier law, namely

$$\mathbf{q} = -K\,\nabla\theta, \qquad K > 0, \tag{1.4}$$

which corresponds to $\kappa(\phi, \theta) = K\theta^2$ (K is a positive real number). In this case, it seems that the only Eq. (1.2) cannot guarantee the positivity of temperature, and thus the existence of a solution to (1.1)–(1.2), for all times. The question under consideration is then whether (1.1)–(1.2) has a global solution (ϕ, θ) satisfying for each $T \in (0, +\infty)$,

$$\theta(x,t) \geq m_T > 0, \qquad (x,t) \in \overline{\Omega} \times [0,T],$$

provided that the initial temperature distribution θ_0 satisfies

$$\theta_0(x) \geq m_0 > 0 \qquad \text{for all } x \in \overline{\Omega}.$$

Though we are not able to give a positive (or negative) answer to the above question, we show in this note that, when the heat flux $\mathbf{q}$ is given by (1.4), it is possible to construct a global in time (weak) solution (ϕ, θ) to (1.1)–(1.2) for some choice of the data, but the temperature distribution θ only satisfies

$$\theta(x,t) > 0 \qquad \text{for almost all } (x,t) \in \Omega \times (0, +\infty).$$

More precisely, we consider the problem

$$\tau\phi_t - \xi^2 \Delta\phi + a(\phi^3 - \phi) = \frac{b}{\theta} \qquad \text{in } \Omega \times (0, +\infty), \tag{1.5}$$

$$c\theta_t - K\Delta\theta = b\phi_t \qquad \text{in } \Omega \times (0, +\infty), \tag{1.6}$$

$$\frac{\partial\phi}{\partial\nu} = 0, \qquad \frac{\partial\theta}{\partial\nu} = 0 \text{ on } \Gamma \times (0, +\infty), \tag{1.7}$$

$$\phi(0) = \phi_0, \qquad \theta(0) = \theta_0 \text{ in } \Omega, \tag{1.8}$$

where

(A1) τ, ξ, a, c, and K are positive real numbers, and $b \in \mathbb{R}$, $b \neq 0$,
(A2) $\phi_0 \in H^1(\Omega)$,
(A3) $\theta_0 \in L^2(\Omega)$ is positive almost everywhere in Ω, and

$$\ln\theta_0 \in L^1(\Omega).$$

We now state our result.

THEOREM 1.1. *Under assumptions* (A1)–(A3), *there exist functions* (ϕ, θ) *such that*, *for each* $T > 0$,

(i) $\phi \in W^{1,2}(0, T, L^2(\Omega)) \cap L^\infty(0, T, H^1(\Omega))$, $\phi(0) = \phi_0$,

(ii) $\theta \in W^{1,2}(0, T, V') \cap L^\infty(0, T, L^2(\Omega)) \cap L^2(0, T, H^1(\Omega))$, $\theta(0) = \theta_0$,

(iii) $\theta > 0$ *a.e. in* Q_T, *and* $1/\theta \in L^1(Q_T)$,

and satisfying

$$\int_0^T \int_\Omega (\tau\phi_t + a(\phi^3 - \phi))\eta + \xi^2\, \nabla\phi . \nabla\eta \, dx\, ds = \int_0^T \int_\Omega \frac{b}{\theta} \eta \, dx\, ds \tag{1.9}$$

for any $\eta \in L^2(0, T, H^1(\Omega)) \cap L^\infty(Q_T)$, *and*

$$c\int_0^T \langle \theta_t, \eta \rangle_{V',V}\, ds + K\int_0^T \int_\Omega \nabla\theta .\, \nabla\eta \, dx\, ds = b\int_0^T \int_\Omega \phi_t\, \eta \, dx\, ds \tag{1.10}$$

for any $\eta \in L^2(0, T, H^1(\Omega))$.

Here, $Q_T = \Omega \times (0, T)$, V' *denotes the dual space of* $V = H^1(\Omega)$, *and* $\langle .,. \rangle_{V',V}$, *the duality pairing between* V *and* V'.

We do not know whether there is uniqueness of solutions to (1.5)–(1.8) in the class described in Theorem 1.1. In fact, we have not been able to obtain more regularity for the inverse temperature $1/\theta$ (which only lies in $L^1(Q_T)$). Any improvement in that direction would certainly open the way to additional results along the directions of positivity of temperature and uniqueness. Also, Theorem 1.1 is not likely to extend to a wider class of nonlinearities s' in (1.1) (at least in space dimension 3, see Remark 2.6 below).

Remark 1.2. The solution (ϕ, θ) to (1.5)–(1.8) we construct in Section 3 actually enjoys the additional property that $\ln \theta \in L^2(0, T, H^1(\Omega))$ for each $T \in (0, +\infty)$.

The remainder of this paper is devoted to the proof of Theorem 1.1: in the next section, we introduce suitable approximations of (1.5)–(1.8), and derive some uniform estimates for their solutions. The key point of this section is an estimate of the inverse temperature in some Orlicz space, which will allow us to pass to the limit in the right-hand side of (1.5) (Lemma 2.5). We may then extract a subsequence that converges to a pair of functions satisfying all the requirements of Theorem 1.1. This is done in the final section.

2. A REGULARISED PROBLEM

Before introducing the regularised problem, we have to find suitable approximations of the initial data (ϕ_0, θ_0). This is done in the next lemma.

LEMMA 2.1. *There exist sequences $(\phi_0^n, \theta_0^n) \in H^2(\Omega, \mathbb{R}^2)$ and a positive constant c_0 depending only on Ω, $|\phi_0|_{H^1(\Omega)}$, $|\theta_0|_{L^2(\Omega)}$, and $|\ln \theta_0|_{L^1(\Omega)}$ satisfying*

$$\lim_{n \to +\infty} \left(|\phi_0^n - \phi_0|_{H^1(\Omega)} + |\theta_0^n - \theta_0|_{L^2(\Omega)} \right) = 0, \tag{2.1}$$

$$\theta_0^n(x) \geq \frac{1}{n}, \qquad \forall x \in \Omega, n \geq 1, \tag{2.2}$$

$$|\phi_0^n|_{H^1(\Omega)} + |\theta_0^n|_{L^2(\Omega)} + |\ln \theta_0^n|_{L^1(\Omega)} \leq c_0, \qquad n \geq 1. \tag{2.3}$$

Proof of Lemma 2.1. The existence of (ϕ_0^n) such that (2.1) and (2.3) hold follows at once from the density of $H^2(\Omega)$ in $H^1(\Omega)$, while the existence of (θ_0^n) such that (2.1)–(2.3) hold is given in the Appendix. ∎

For each integer $n \geq 1$, we consider the following initial-boundary value problem:

$$\tau\,\phi_t^n - \xi^2\,\Delta\phi^n + a\big((\phi^n)^3 - \phi^n\big) = \frac{b}{\theta^n} \qquad \text{in } \Omega \times (0, +\infty), \tag{2.4}$$

$$c\,\theta_t^n - K\,\Delta\left(\theta^n - \frac{1}{n\theta^n}\right) - \frac{1}{n\theta^n} = b\,\phi_t^n \qquad \text{in } \Omega \times (0, +\infty), \tag{2.5}$$

$$\frac{\partial\phi^n}{\partial\nu} = 0, \qquad \frac{\partial}{\partial\nu}\left(\theta^n - \frac{1}{n\theta^n}\right) = 0 \text{ on } \Gamma \times (0, +\infty), \tag{2.6}$$

$$\phi^n(0) = \phi_0^n, \qquad \theta^n(0) = \theta_0^n \text{ in } \Omega. \tag{2.7}$$

PROPOSITION 2.2. *For each integer $n \geq 1$, the initial-boundary value problem* (2.4)–(2.7) *has a unique classical solution* (ϕ^n, θ^n) *with* $\theta^n > 0$; *more precisely,*

$$(\phi^n, \theta^n) \in \mathscr{C}\big(\overline{\Omega} \times [0, +\infty), \mathbb{R} \times (0, +\infty)\big) \cap \mathscr{C}^{2,1}\big(\overline{\Omega} \times (0, +\infty), \mathbb{R}^2\big).$$

The proof of Proposition 2.2 is similar to that of [5, Proposition 4; 12, Sect. 3] and we omit it here. The local existence in time and uniqueness of the classical solution to (2.4)–(2.7) follow from the general theory of H. Amann [1]. Global existence is then a consequence of [1] and L^∞-estimates for ϕ^n, θ^n, and $1/\theta^n$ which are derived in the same way as in [12, Sect. 3; 5, Sect. 3]. The main difference being worth mentioning here occurs in the proof of the counterpart of [5, Lemma 6]. Indeed, one shall differentiate (2.4) with respect to time and multiply it by (ϕ_t^n/n) (instead of ϕ_t^n), multiply (2.5) by $(\theta^n - 1/(n\theta^n))_t$ (instead of $(-1/\theta^n)_t$), and integrate the sum of the resulting identities over $\Omega \times (0, t)$ to obtain estimates similar to those of [5, Lemma 6]. Notice also that it follows from Proposition 2.2 that for each integer $n \geq 1$ and $T > 0$, there exists a positive real number $\alpha(n, T)$ such that

$$0 < \alpha(n, T) \leq \theta^n(x, t), \qquad (x, t) \in \overline{\Omega} \times [0, T].$$

We now derive some estimates satisfied by (ϕ^n, θ^n) uniformly with respect to $n \geq 1$. In the following, we fix a positive real number T, and denote by C_T and $\overline{C}_T$ any positive constant depending only on $\Omega, \tau, \xi, a, b, c, K, |\phi_0|_{H^1(\Omega)}, |\theta_0|_{L^2(\Omega)}, |\ln\,\theta_0|_{L^1(\Omega)}$, and T.

LEMMA 2.3.

$$|\phi^n|_{L^\infty(0,T,H^1(\Omega))} + |\phi_t^n|_{L^2(Q_T)} + |\theta^n|_{L^\infty(0,T,L^1(\Omega))} + \frac{1}{\sqrt{n}}\left|\frac{1}{\theta^n}\right|_{L^2(0,T,H^1(\Omega))}$$
$$+ |\ln\,\theta^n|_{L^\infty(0,T,L^1(\Omega))} + |\ln\,\theta^n|_{L^2(0,T,H^1(\Omega))} \leq C_T. \tag{2.8}$$

Proof of Lemma 2.3. We take the scalar product in $L^2(\Omega)$ of (2.4) with ϕ_t^n, of (2.5) with $(1 - 1/\theta^n)$, and add both; this gives

$$\frac{d}{dt}\int_\Omega \left(\frac{\xi^2}{2}|\nabla\phi^n|^2 + \frac{a}{4}\left((\phi^n)^2 - 1\right)^2 + c\,\theta^n - c\ln\theta^n \right) dx$$

$$+ \tau\int_\Omega |\phi_t^n|^2\,dx + K\int_\Omega \left(|\nabla\ln\theta^n|^2 + \frac{1}{n}\left|\nabla\left(\frac{1}{\theta^n}\right)\right|^2 \right) dx + \frac{1}{n}\int_\Omega \left(\frac{1}{\theta^n}\right)^2 dx$$

$$= b\int_\Omega \phi_t^n\,dx + \frac{1}{n}\int_\Omega \frac{1}{\theta^n}\,dx$$

$$\leq \frac{\tau}{2}\int_\Omega |\phi_t^n|^2\,dx + \frac{1}{2n}\int_\Omega \left(\frac{1}{\theta^n}\right)^2 dx + C_T,$$

thanks to the Young inequality. After integration over $(0,t)$, $t \in (0,T)$, we infer from the Poincaré inequality, the continuous embedding of $H^1(\Omega)$ in $L^4(\Omega)$, (2.3), and the above inequality that

$$\int_\Omega \left(\frac{\xi^2}{2}|\nabla\phi^n(t)|^2 + \frac{a}{4}\left((\phi^n(t))^2 - 1\right)^2 + c\,\theta^n(t) - c\ln\theta^n(t) \right) dx$$

$$+ \int_0^t \int_\Omega \left(\frac{\tau}{2}|\phi_t^n|^2 + K|\nabla\ln\theta^n|^2 \right) dx\,ds + \frac{\overline{C}_T}{n}\int_0^t \left|\frac{1}{\theta^n}\right|_{H^1(\Omega)}^2 ds \leq C_T. \tag{2.9}$$

We also have

$$\int_\Omega |\ln\theta^n(t)|\,dx = -\int_\Omega \ln\theta^n(t)\,dx + 2\int_{\{\theta^n(t)\geq 1\}} \ln\theta^n(t)\,dx$$

$$\int_\Omega |\ln\theta^n(t)|\,dx \leq \int_\Omega \left(\tfrac{1}{2}\theta^n(t) - \ln\theta^n(t)\right) dx + C_T. \tag{2.10}$$

Combining (2.9) and (2.10), and using the Young inequality yield

$$\int_\Omega \left(|\nabla\phi^n(t)|^2 + |\phi^n(t)|^4 + \theta^n(t) + |\ln\theta^n(t)| \right) dx$$

$$+ \int_0^t \int_\Omega \left(|\phi_t^n|^2 + |\nabla\ln\theta^n|^2 \right) dx\,ds + \frac{1}{n}\int_0^t \left|\frac{1}{\theta^n}\right|_{H^1(\Omega)}^2 ds \leq C_T.$$

Now, (2.8) follows from the Poincaré inequality and the above inequality. ∎

We next derive estimates for θ^n.

LEMMA 2.4.

$$|\theta^n|_{L^\infty(0,T,L^2(\Omega))} + |\theta^n|_{L^2(0,T,H^1(\Omega))} + |\theta^n_t|_{L^2(0,T,V')} \le C_T. \qquad (2.11)$$

Proof of Lemma 2.4. Let $t \in (0,T)$, and take the scalar product in $L^2(\Omega \times (0,t))$ of (2.5) with θ^n; this gives

$$\frac{c}{2}\int_\Omega |\theta^n(t)|^2\,dx + K\int_0^t\int_\Omega \left(|\nabla\theta^n|^2 + \frac{1}{n}\left|\frac{\nabla\theta^n}{\theta^n}\right|^2\right)dx\,ds$$
$$\le \frac{c}{2}\int_\Omega |\theta^n_0|^2\,dx + b\int_0^t\int_\Omega \phi^n_t\theta^n\,dx\,ds + C_T.$$

We now use (2.3), the Young inequality, and (2.8), and find

$$\frac{c}{2}\int_\Omega |\theta^n(t)|^2\,dx + K\int_0^t\int_\Omega |\nabla\theta^n|^2\,dx\,ds \le C_T + C_T\int_0^t\int_\Omega |\theta^n|^2\,dx\,ds.$$

Using the Gronwall lemma, we obtain

$$|\theta^n|_{L^\infty(0,T,L^2(\Omega))} + |\theta^n|_{L^2(0,T,H^1(\Omega))} \le C_T. \qquad (2.12)$$

Now, (2.11) follows from (2.8), (2.12), and (2.5). ∎

We finally derive an estimate for the inverse temperature.

LEMMA 2.5.

$$\left|\frac{1}{\theta^n}\ln\left(\frac{1}{\theta^n}\right)\right|_{L^1(Q_T)} \le C_T. \qquad (2.13)$$

Proof of Lemma 2.5. We first infer from (2.8) that

$$|(-\ln\theta^n)^+|_{L^2(0,T,H^1(\Omega))} \le C_T, \qquad (2.14)$$

where z^+ denotes the positive part of z.

It follows from (2.4) that

$$|b|\int_0^T\int_\Omega \frac{(-\ln\theta^n)^+}{\theta^n}\,dx\,ds \le \tau|\ln\theta^n|_{L^2(Q_T)}|\phi^n_t|_{L^2(Q_T)}$$
$$+ \xi^2|\nabla\phi^n|_{L^2(Q_T)}|\nabla(-\ln\theta^n)^+|_{L^2(Q_T)}$$
$$+ a|\ln\theta^n|_{L^2(Q_T)}\left(|\phi^n|^3_{L^6(Q_T)} + |\phi^n|_{L^2(Q_T)}\right).$$

We then infer from (2.8), (2.14), and the continuity of the embedding of $H^1(\Omega)$ in $L^6(\Omega)$ that

$$\int_0^T \int_\Omega \frac{(-\ln \theta^n)^+}{\theta^n}\,dx\,ds \le C_T. \tag{2.15}$$

It now follows from (2.8) and (2.15) that

$$\begin{aligned}
\left| \frac{1}{\theta^n} \ln\left(\frac{1}{\theta^n}\right) \right|_{L^1(Q_T)} &\le \int_{\{\theta^n \ge 1\}} \frac{\ln \theta^n}{\theta^n}\,dx\,ds + \int_{\{\theta^n \le 1\}} \frac{-\ln \theta^n}{\theta^n}\,dx\,ds \\
&\le \int_{\{\theta^n \ge 1\}} |\ln \theta^n|\,dx\,ds + \int_{\{\theta^n \le 1\}} \frac{(-\ln \theta^n)^+}{\theta^n}\,dx\,ds \\
&\le |\ln \theta^n|_{L^1(Q_T)} + \int_0^T \int_\Omega \frac{(-\ln \theta^n)^+}{\theta^n}\,dx\,ds \\
&\le C_T,
\end{aligned}$$

hence (2.13). ∎

Remark 2.6. As already mentioned in the Introduction, the estimate (2.13) is the key point of the proof of Theorem 1.1; since it provides the uniform integrability of the sequence $(1/\theta^n)_{n\ge 1}$ in $L^1(Q_T)$ we will use it to pass to the limit in the right-hand side of (2.4). Notice that in order for (2.13) to hold true, we need to be able to control each term of the left-hand side of (2.4), including $a((\phi^n)^3 - \phi^n)$ which corresponds to $s'(\phi^n)$ with the notations of (1.1)–(1.2). Consequently, the above method does not seem to extend to the general case (1.1)–(1.2) when $s'(\phi) = a(\phi^3 - \phi)$ is replaced by either a smooth function s' growing faster than ϕ^3 or by singular or multivalued operators s' such as (see, e.g., [10, 6, 7])

$$s'(\phi) = \ln\left(\frac{1+\phi}{1-\phi}\right) - 2\phi, \qquad \phi \in (-1,1),$$

or

$$s'(\phi) = \begin{cases} [-2, +\infty) & \text{if } \phi = 1, \\ -2\phi & \text{if } \phi \in (-1,1), \\ (-\infty, 2] & \text{if } \phi = -1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Nevertheless, a result similar to Theorem 1.1 is obtained in [9] for (1.5)–(1.8) when the right-hand side of (1.5) and (1.6) are replaced by

$\lambda'(\phi)/\theta$ and $(\lambda(\phi))_t$, respectively, for some concave function λ with $\lambda' \in W^{1,\infty}(\mathbb{R})$.

3. PROOF OF THEOREM 1.1

Let T be a positive real number. We first infer from (2.8) and [11, Corollary 4] that

$$(\phi^n) \text{ is relatively compact in } \mathscr{C}\big([0,T], L^2(\Omega)\big). \tag{3.1}$$

We also infer from (2.11) and [11, Corollary 4] that

$$(\theta^n) \text{ is relatively compact in } \mathscr{C}([0,T], V') \text{ and in } L^2(Q_T). \tag{3.2}$$

It now follows from (2.8), (2.11), (3.1), and (3.2) that there is a subsequence of (ϕ^n, θ^n) (which we will still denote by (ϕ^n, θ^n)) and functions (ϕ, θ, l) satisfying

$$\begin{aligned}
&\phi \in W^{1,2}\big(0,T, L^2(\Omega)\big) \cap L^\infty\big(0,T, H^1(\Omega)\big),\\
&\theta \in W^{1,2}(0,T, V') \cap L^\infty\big(0,T, L^2(\Omega)\big) \cap L^2\big(0,T, H^1(\Omega)\big),\\
&l \in L^2\big(0,T, H^1(\Omega)\big),
\end{aligned}$$

and such that

$$\phi^n \to \phi \qquad \text{in } \mathscr{C}\big([0,T], L^2(\Omega)\big) \text{ and a.e. in } Q_T, \tag{3.3}$$

$$\theta^n \to \theta \qquad \text{in } \mathscr{C}([0,T], V'), \text{ in } L^2(Q_T) \text{ and a.e. in } Q_T, \tag{3.4}$$

$$\ln \theta^n \rightharpoonup l \qquad \text{in } L^2(Q_T). \tag{3.5}$$

Since $H^1(\Omega)$ is compactly embedded in $L^4(\Omega)$, (2.8) and (3.3) yield that (ϕ^n) converges to ϕ in $L^4(Q_T)$, hence

$$(\phi^n)^3 \to \phi^3 \qquad \text{in } L^{4/3}(Q_T). \tag{3.6}$$

Next, since $r \mapsto \ln r$ is a maximal monotone graph in $\mathbb{R}$ with domain $(0, +\infty)$, we infer from (3.4), (3.5), and [2] that

$$\theta > 0 \qquad \text{and} \qquad l = \ln\theta \text{ a.e. in } Q_T. \tag{3.7}$$

Therefore, on the one hand, it follows from (3.4) and (3.7) that

$$\frac{1}{\theta^n} \to \frac{1}{\theta} \qquad \text{a.e. in } Q_T. \tag{3.8}$$

On the other hand, since $\lim_{r\to+\infty}\ln r = +\infty$, we infer from (2.13) that $(1/\theta^n)$ is uniformly integrable (see, e.g., [4, p. 38]). This last fact and (3.8) ensure that

$$\frac{1}{\theta} \in L^1(Q_T)$$

and that

$$\frac{1}{\theta^n} \to \frac{1}{\theta} \qquad \text{in } L^1(Q_T) \tag{3.9}$$

(see, e.g., [4, p. 36]).

Finally, it follows from (2.8) that

$$\frac{1}{n\theta^n} \to 0 \qquad \text{in } L^2\big(0,T,H^1(\Omega)\big). \tag{3.10}$$

We are now ready to complete the proof of Theorem 1.1: indeed, (1.9) follows at once from (2.4), (2.8), (3.3), (3.6), and (3.9); and (1.10) is a straightforward consequence of (2.5), (2.8), (2.11), (3.4), and (3.10). In addition, (3.3) and (2.1) ensure that $\phi(0) = \phi_0$, while (3.4) and (2.1) ensure that $\theta(0) = \theta_0$. Finally, the regularity results stated in Theorem 1.1 follow from the above analysis. Since T is arbitrary, the proof is complete. ∎

APPENDIX: APPROXIMATION OF θ_0

LEMMA A.1. *Let U be an open bounded subset of $\mathbb{R}^m, m \geq 1$, and consider a function $z \in L^2(U)$ which is positive almost everywhere in U and such that $\ln z$ belongs to $L^1(U)$. Then, there exist a sequence $(z_n)_{n\geq 1}$ of functions of $H^2(U)$ and a positive constant γ depending on U, $|z|_{L^2(U)}$, and $|\ln z|_{L^1(U)}$ satisfying*

$$\lim_{n\to+\infty} |z_n - z|_{L^2(U)} = 0, \tag{A.1}$$

$$\frac{1}{n} \leq z_n(x), \qquad x \in U, \tag{A.2}$$

$$|\ln z_n|_{L^1(U)} \leq \gamma. \tag{A.3}$$

Proof of Lemma A.1. We put

$$Z_1 = \min\{z, 1\}, \qquad Z_2 = z - Z_1.$$

We now consider an integer $n \geq 1$. Since U has finite Lebesgue measure and $Z_1 \in L^1(U)$, we infer from the Lusin theorem that there is a function $w_n \in \mathscr{C}(U)$ with compact support in U such that

$$\operatorname{meas}\bigl(\{x \in U, Z_1(x) \neq w_n(x)\}\bigr) \leq \frac{1}{n}, \tag{A.4}$$

$$0 \leq w_n \leq 1. \tag{A.5}$$

We put

$$U_n = \{x \in U, Z_1(x) = w_n(x)\}.$$

Since w_n is continuous and compactly supported in U, we may extend it by zero to a continuous and compactly supported function $\overline{w}_n$ of $\mathbb{R}^m$. Now, using a convolution with a sequence of mollifiers, we may find a $\mathscr{C}^\infty$-smooth function W_n such that

$$|W_n - \overline{w}_n|_{\mathscr{C}(\overline{U})} \leq \frac{1}{2n}, \tag{A.6}$$

$$0 \leq W_n \leq 1. \tag{A.7}$$

We put

$$z_{1,n}(x) = W_n(x) + \frac{1}{n}, \qquad x \in U.$$

Since U is bounded, $z_{1,n} \in H^2(U)$, and we infer from (A.7) that

$$\frac{1}{n} \leq z_{1,n} \leq 2. \tag{A.8}$$

It follows from (A.6), (A.5), and (A.4) that

$$\begin{aligned} |z_{1,n} - Z_1|_{L^1(U)} &\leq \frac{\operatorname{meas}(U)}{n} + |W_n - w_n|_{L^1(U)} + \int_{U \setminus U_n} |w_n - Z_1|\, dx, \\ &\leq \frac{\operatorname{meas}(U)}{n} + \frac{\operatorname{meas}(U)}{2n} + 2\operatorname{meas}(U \setminus U_n), \\ |z_{1,n} - Z_1|_{L^1(U)} &\leq \frac{\gamma}{n}. \end{aligned}$$

The Hölder inequality and (A.8) then yield

$$|z_{1,n} - Z_1|_{L^2(U)} \leq \frac{\gamma}{\sqrt{n}}. \tag{A.9}$$

It next follows from (A.6) that

$$\forall x \in U_n, \qquad z_{1,n}(x) \geq Z_1(x). \tag{A.10}$$

Combining (A.8), (A.10), and (A.4), we obtain

$$\begin{aligned} \|\ln z_{1,n}\|_{L^1(U)} &\leq \operatorname{meas}(U)\ln(2) - \int_{U_n} \ln\left(\frac{Z_1}{2}\right) dx + \int_{U\setminus U_n} \ln(2n)\, dx, \\ &\leq \gamma + \|\ln Z_1\|_{L^1(U)} + \frac{\ln(2n)}{n}, \end{aligned}$$

$$\|\ln z_{1,n}\|_{L^1(U)} \leq \gamma. \tag{A.11}$$

Now, since Z_2 is nonnegative, a standard procedure yields the existence of a nonnegative function $z_{2,n} \in H^2(U)$ such that

$$|z_{2,n} - Z_2|_{L^2(U)} \leq \frac{1}{n}. \tag{A.12}$$

We put $z_n = z_{1,n} + z_{2,n}$, which lies in $H^2(U)$. Since $z_{2,n}$ is nonnegative, (A.2) follows from (A.8). Next (A.1) is a straightforward consequence of (A.9) and (A.12). Finally, since $z_{2,n}$ is nonnegative, we have

$$z_n \geq z_{1,n}. \tag{A.13}$$

We then infer from (A.13), (A.11), and (A.1) that

$$\begin{aligned} \|\ln z_n\|_{L^1(U)} &\leq -\int_{\{z_n \leq 1\}} \ln z_{1,n}\, dx + \int_{\{z_n \geq 1\}} \ln z_n\, dx \\ &\leq \|\ln z_{1,n}\|_{L^1(U)} + |z_n|_{L^1(U)}, \end{aligned}$$

hence (A.3). ∎

ACKNOWLEDGMENT

I thank the referees for their valuable comments on the first version of this paper.

REFERENCES

1. H. Amann, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, *in* "Function Spaces, Differential Operators and Nonlinear Analysis" (H. Triebel and H. J. Schmeisser, Eds.), Teubner-Texte Math., Vol. **133**, pp. 9–126, Teubner, Stuttgart, 1993.

2. H. Brezis, "Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert," North-Holland, Amsterdam, 1973.
3. P. Colli and Ph. Laurençot, Weak Solutions of the Penrose–Fife Phase-Field Model for a Class of Admissible Heat Flux Laws, *Phys. D*, to appear.
4. C. Dellacherie and P. A. Meyer, "Probabilités et potentiel," Chaps. I–IV, Hermann, Paris, 1975.
5. W. Horn, Ph. Laurençot, and J. Sprekels, Global solutions to a Penrose–Fife phase-field model under flux boundary conditions for the inverse temperature, *Math. Methods Appl. Sci.* **19** (1996), 1053–1072.
6. N. Kenmochi and M. Niezgódka, Systems of nonlinear parabolic equations for phase change problems, *Adv. Math. Sci. Appl.* **3** (1993), 89–117.
7. Ph. Laurençot, Solutions to a Penrose–Fife model of phase-field type, *J. Math. Anal. Appl.* **185** (1994), 262–274.
8. Ph. Laurençot, Weak solutions to a Penrose–Fife model for phase transitions, *Adv. Math. Sci. Appl.* **5** (1995), 117–138.
9. Ph. Laurençot, Weak solutions to the Penrose–Fife model with Fourier law and quadratic nonlinearity, manuscript.
10. O. Penrose and P. C. Fife, Thermodynamically consistent models of phase-field type for the kinetics of phase transitions, *Phys. D* **43** (1990), 44–62.
11. J. Simon, Compact sets in the space $L^p(0, T; B)$, *Ann. Mat. Pura Appl.* **146** (1987), 65–96.
12. J. Sprekels and S. Zheng, Global smooth solutions to a thermodynamically consistent model of phase-field type in higher space dimensions, *J. Math. Anal. Appl.* **176** (1993), 200–223.
13. S. Zheng, Global existence for a thermodynamically consistent model of phase-field type, *Differential Integral Equations* **5** (1992), 241–253.